

A remark on the Koide relation for quarks

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The charged lepton masses obey to high precision the so-called Koide relation. We propose a generalization of this relation to quarks. It includes up and down quarks of the three generations and is numerically reasonably close to the Koide limit.

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The pole masses of three charged leptons have been measured to an unprecedented precision [1]:

$$m_e = 0.510998910 \pm 0.000000013 \text{ MeV}, \quad (1a)$$

$$m_\mu = 105.658367 \pm 0.000004 \text{ MeV}, \quad (1b)$$

$$m_\tau = 1776.82^{+0.16}_{-0.26} \text{ MeV}. \quad (1c)$$

To a surprising degree of accuracy they obey the empirical Koide mass relation [2, 3]:

$$K_\ell \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}. \quad (2)$$

Defining a deviation from the Koide limit, $\Delta K_\ell \equiv (3K_\ell/2 - 1)$, and substituting the experimental charged lepton masses into (2) we find $-2 \cdot 10^{-5} < \Delta K_\ell < 9 \cdot 10^{-7}$. It is also astonishing that the electron mass, being orders of magnitude smaller than the muon and tau-lepton masses, still plays an important role in this relation. Neglecting m_e in (2) we would find $\Delta K_\ell \sim 2 \cdot 10^{-2}$.

There have been quite a few attempts to derive (2) from flavor symmetries or grand-unified extensions of the Standard Model [4–6]. Inspired by the idea of grand unification, one may wonder whether a similar mass relations also exist for quarks. Using current values of d , s and b -quark masses [1]:

$$m_d = 4.1 - 5.7 \text{ MeV}, \quad (3a)$$

$$m_s = 100^{+30}_{-20} \text{ MeV}, \quad (3b)$$

$$m_b = 4.19^{+0.18}_{-0.06} \text{ GeV}, \quad (3c)$$

and defining, analogously to (2),

$$K_d \equiv \frac{m_d + m_s + m_b}{(\sqrt{m_d} + \sqrt{m_s} + \sqrt{m_b})^2}, \quad (4)$$

we find $K_d \sim 0.72$, i.e. a considerable deviation from the Koide limit with $\Delta K_d \sim 8 \cdot 10^{-2}$. For up-type quarks the mass spectrum has a stronger hierarchy than for down-type quarks [1]:

$$m_u = 1.7 - 3.1 \text{ MeV}, \quad (5a)$$

$$m_c = 1.29^{+0.05}_{-0.11} \text{ GeV}, \quad (5b)$$

$$m_t = 172.9 \pm 0.6 \pm 0.9 \text{ GeV}, \quad (5c)$$

and therefore the analogue of (2) for up-type quarks,

$$K_u \equiv \frac{m_u + m_c + m_t}{(\sqrt{m_u} + \sqrt{m_c} + \sqrt{m_t})^2}, \quad (6)$$

deviates from the Koide limit even stronger, $K_u \sim 0.85$, which corresponds to $\Delta K_u \sim 27 \cdot 10^{-2}$. One could argue, that due to the non-perturbative nature of quantum chromodynamics at low energies, the pole masses of light quarks are not well defined and one should make use of the running quark masses at the scale $M_Z = 91.2 \text{ GeV}$ [7]. The latter are given by [8]:

$$m_u = 1.27^{+0.50}_{-0.42} \text{ MeV}, \quad (7a)$$

$$m_d = 2.90^{+1.24}_{-1.19} \text{ MeV}, \quad (7b)$$

$$m_s = 55^{+16}_{-15} \text{ MeV}, \quad (7c)$$

$$m_b = 0.619 \pm 0.084 \text{ GeV}, \quad (7d)$$

$$m_c = 2.89 \pm 0.09 \text{ GeV}, \quad (7e)$$

$$m_t = 171.7 \pm 3.0 \text{ GeV}. \quad (7f)$$

Substituting (7) into (4) and (6) we find $K_d \sim 0.74$ and $K_u \sim 0.89$ respectively. This corresponds to $\Delta K_d \sim 11 \cdot 10^{-2}$ and $\Delta K_u \sim 33 \cdot 10^{-2}$. That is, the running effects induce an even stronger deviation from the Koide limit. The same also applies to leptons. The running charged lepton masses at the scale $M_Z = 91.2 \text{ GeV}$ are given by [8]:

$$m_e = 0.486570161 \pm 0.000000042 \text{ MeV}, \quad (8a)$$

$$m_\mu = 102.7181359 \pm 0.00000092 \text{ MeV}, \quad (8b)$$

$$m_\tau = 1746.24^{+0.20}_{-0.19} \text{ MeV}. \quad (8c)$$

Substituting these values into the Koide relation for leptons (2) we find $1.87 \cdot 10^{-3} < \Delta K_\ell < 1.91 \cdot 10^{-3}$. This implies in particular, that the running masses at high scales are not compatible with the exact Koide limit, $K_\ell = 2/3$. This result is somewhat counterintuitive. If there is an underlying high-scale symmetry which ensures (2) for leptons, one could expect that the deviation from the Koide limit would be smaller at high scales.

It was noted in [7] that one could divide quarks into light and heavy ones, instead of up- and down-like quarks, that is, according to their mass instead of the isospin. The corresponding Koide parameters are defined by:

$$K_{light} \equiv \frac{m_u + m_d + m_s}{(\sqrt{m_u} + \sqrt{m_d} + \sqrt{m_s})^2}, \quad (9a)$$

$$K_{heavy} \equiv \frac{m_c + m_b + m_t}{(\sqrt{m_c} + \sqrt{m_b} + \sqrt{m_t})^2}. \quad (9b)$$

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Substituting (3) and (5) into (9) we find a rather good agreement with the Koide limit for the heavy quarks, $-5 \cdot 10^{-3} < \Delta K_{heavy} < 1 \cdot 10^{-2}$, and a deviation from it for the light ones, $-20 \cdot 10^{-2} < \Delta K_{light} < -6 \cdot 10^{-2}$. If we take the running masses at the scale $M_Z = 91.2$ GeV instead, we will obtain $5 \cdot 10^{-2} < \Delta K_{heavy} < 9 \cdot 10^{-2}$ and $-20 \cdot 10^{-2} < \Delta K_{light} < -2 \cdot 10^{-2}$ respectively.

Although dividing quarks into the light and the heavy ones numerically gives a rather good agreement with the Koide relation, the fact that s and c quarks – components of the same $SU_L(2)$ doublet – enter the expressions for K_{light} and K_{heavy} separately is counterintuitive. A somewhat more natural generalization of the Koide relation (2) to quarks is an expression which sums over up and down components of the three generations:

$$K_q \equiv \frac{\sum m_q}{(\sum \sqrt{m_q})^2}. \quad (10)$$

Since the active neutrino masses are very small, a similar generalization of the Koide relations for leptons would not affect the range for ΔK_ℓ indicated above. Substituting (3) and (5) into (10) we find for the deviation from the Koide limit $-5 \cdot 10^{-2} < \Delta K_q < -4 \cdot 10^{-2}$. If we take the running quark masses at the scale $M_Z = 91.2$ GeV instead, we will obtain $2 \cdot 10^{-2} < \Delta K_q < 5 \cdot 10^{-2}$. It is interesting that due to the running effects ΔK_q crosses zero. That is, at some scale $2 \text{ GeV} < \mu < M_Z$ the quark masses satisfy (within the uncertainties) the Koide relation $K_q = 2/3$.

It was noted in [9] that the Koide relation has a geometric interpretation. Let $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ define a vector in a three-dimensional vector space and θ_ℓ be the angle between this vector and the vector $(1, 1, 1)$,

$$\theta_\ell \equiv \arccos \frac{(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}) \cdot (1, 1, 1)}{|(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})| |(1, 1, 1)|}. \quad (11)$$

Then the Koide relation (2) is equivalent to the statement $\theta_\ell = \pi/4$. Any deviation of K from the Koide limit, $K = 2/3$, implies also a deviation from $\theta = \pi/4$. Thus, θ_d and θ_s substantially deviate from $\pi/4$, whereas θ_{light} and θ_{heavy} are close to this limit. It is interesting that if we define an analogue of (11) for quarks in a six-dimensional vector space,

$$\theta_q \equiv \arccos \frac{(\sqrt{m_d}, \sqrt{m_u}, \sqrt{m_s}, \dots) \cdot (1, 1, 1, \dots)}{|(\sqrt{m_d}, \sqrt{m_u}, \sqrt{m_s}, \dots)| |(1, 1, 1, \dots)|}, \quad (12)$$

then we will find that this angle is again a rational multiple of π , $\theta_q \approx \pi/3$.

To summarize, generalizations of the original Koide relation to quarks which include only the up-type or only the down-type quarks show a considerable deviation from the Koide limit $K = 2/3$. In this letter we have proposed a generalization which includes up and down quarks of the three generations. Interestingly enough, although masses of the light (u , d and s) quarks are much smaller than masses of the heavy ones (c , b and t), their inclusion into the sum makes it substantially closer to the Koide limit. This resembles the role played by the electron mass in the Koide relation for charged leptons. Furthermore, similarly to the Koide relation for leptons, the proposed relation for quarks also allows for a geometric interpretation: the angle between vectors $(\sqrt{m_d}, \sqrt{m_u}, \sqrt{m_s}, \dots)$ and $(1, 1, 1, \dots)$ in a six-dimensional vector space is very close to $\pi/3$.

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